ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 4, pp. 373–390.

Composite Observer of a Linear Time-Varying Singularly Perturbed System with Quasidifferentiable Coefficients

O. B. Tsekhan

Yanka Kupala State University of Grodno, Grodno, Belarus
e-mail: tsekhan@grsu.by

Received November 28, 2023
Revised February 2, 2024
Accepted March 4, 2024

Abstract—For a linear time-varying singularly perturbed system with a small parameter μ for a part of derivatives and quasi-differentiable coefficients, existence conditions are established and μ-asymptotic composite full- and reduced-order observers are constructed. The error in estimating a state with an arbitrary predetermined exponential decay rate converges to an infinitesimal value of the same order of smallness as the small parameter. The observer gain vector are expressed in terms of the gain vectors of subsystems of smaller dimension than the original one and independent of the small parameter, and the parameters of the original system are subject to weaker requirements than those previously known. A constructive algorithm for calculating the gain vector of a composite observer is presented.

Keywords: : robust observer, time-varying singularly perturbed system, quasi-differentiability

DOI: 10.31857/S0005117924040022

1. INTRODUCTION

The problem of estimating the states of dynamic systems using available information is relevant due to its importance for various positioning systems (location determination) of control objects. However, in real situations, direct measurement of the state vector may be difficult (for cost reasons, due to technological limitations, etc.). In this case, states can be estimated using a specially constructed dynamic system called an observer (estimator). The observer's input is the output function of the original system, and its state must, in one sense or another, approximate the state of the original system [1–6]. If, with increasing time, the state of the observer converges to the phase vector of the system, then the observer is called *asymptotic*. If, in addition, there is exponential convergence, then such an observer is called *exponential*.

The definition of an observer was first introduced by Luenberger in his dissertation in 1963 (see also [7, 8]). He showed that for every linear system being observed, an observer that is itself a linear system can be designed with the estimation error tending to zero at a given rate. In this case, the task of constructing an observer comes down to choosing the gain, which is calculated from the system parameters and does not depend on the output [8].

Singularly perturbed systems (SPS) with a small parameter μ for a part of derivatives are widely used in applications in aviation, chemistry, electrical engineering, mechanics, etc. as models of multi-tempo processes when simulating the dynamics of aircraft, chemical reactions, movements of robotic manipulators, etc. (see reviews [9–13] and references there). For SPS, depending on the information about the small parameter and the needs of the applications, different formulations of observation problems can be considered: with a known value of the small parameter μ [14], for a known closed interval of values of the small parameter $\mu \in [\underline{\mu}, \overline{\mu}] \subset (0, \mu^0]$ [15], for an unknown value of the small parameter μ [16]. Formulations are also distinguished depending on the composition of the components being evaluated: estimation of both slow and fast components, or only slow components (see [10] and references there). In applications, the exact value of the small parameter μ can be unknown. Therefore, it is important that observers provide a "good" state estimate for the entire μ-parametric family of SPSs for various values of the parameter μ. Observers provide estimates of the system states in conditions where the system model is not precisely known, are called *robust* observers.

Depending on the dimension of the observer *full-order observers*, the state of which has the same dimension as the state of the observed system, as well as *reduced-order observers*, the dimension of whose states is smaller (by the dimension of the output), are distinguished [1, p. 379]. Identification in the SPS of fast (parasitic) dynamics, when describing which a small parameter appears in the model, allows us to further reduce observers.

The use of the multi-tempo structure of the SPS allows, when constructing observers, to operate with systems of a smaller dimension than the original one (see [14, 16, 17])—subsystems of slow and fast movements separated by time scales. In this case, the gain for the observer of the original SPS can be calculated as a composition of the gains of separately designed observers of the slow and fast subsystems. This approach for constructing a composite [14] observer is used in [17] for linear time-invariant SPS (LTISPS). In [16] for the LTISPS the concept was introduced and the construction of μ-asymptotic observers was justified, for which the state estimation error with an arbitrary predetermined exponential decay rate converges to an infinitesimal value of the same order of smallness as the small parameter. In [14] the construction of a μ-asymptotic full-order observer for a linear time-varying SPS (LTVSPS) is described, but the rules for constructing such observers are not given there. Research on the design of observers of slow states of nonlinear SPS can be found in [18–21] and the works cited there.

Many real dynamic systems are described by models whose parameters depend on time. Constructive methods for the analysis and synthesis of time-varying systems can be obtained for systems that can be reduced to time-invariant [22].Time-varying systems can arise during the linearization of time-invariant nonlinear systems. Linearization of time-invariant systems can lead to a decrease in the smoothness of the system parameters. The use of the quasi-differentiation apparatus [23, 24] allows us to expand the class of time-varying systems for which it is possible to obtain constructive results.

When we consider deterministic observation systems, state estimation assumes that the system has a certain type of observability. For a time-invariant system, complete observability guarantees the existence of an asymptotic observer [2]. For a time-varying system, its uniformly complete observability is required. However, this property is difficult to verify in terms of the coefficients of the original observation system, and therefore, from a constructive point of view, it is not very effective. The approach proposed in [5] based on the quasi-differentiation technique makes it possible to constructively construct observers for uniformly observed time-varying systems, while weakening the known requirements for smoothness of coefficients.

Issues of SPS observability were previously studied by the author in [25–30]. The contribution of this work is that, in contrast to [25–30], the problem of constructing observers is studied; in contrast to [5], a singularly perturbed system is considered; in comparison with [14], a constructive algorithm for constructing a composite exponential LTVSPS observer has been developed, while the requirements for the smoothness of the system parameters have been relaxed; in contrast to [16, 17], time-varying SPS is considered.

2. FORMULATION OF THE PROBLEM

We consider a linear time-varying singularly perturbed system (LTVSPS):

$$\begin{aligned} &\dot{x}(t) = A_1(t)x(t) + A_2(t)y(t), \quad x \in \mathbb{R}^{n_1}, \quad y \in \mathbb{R}^{n_2}, \\ &\mu\dot{y}(t) = A_3(t)x(t) + A_4(t)y(t), \quad t \in T = [t_0, +\infty), \end{aligned} \tag{1}$$

with scalar output

$$v(t) = c_1(t)x(t) + c_2(t)y(t), \quad v \in \mathbb{R}, \quad t \in T = [t_0, +\infty). \tag{2}$$

Here μ is a small parameter, $\mu \in (0, \mu^0]$, $\mu^0 \ll 1$, $x(t)$ is an unknown vector of slow variables, $y(t)$ is an unknown vector of fast variables, $A_i(t)$, $i = \overline{1,4}$, $c_j(t)$, $j = 1, 2$ are continuous matrix functions of compartible orders and row vector functions bounded on T, respectively, $v(t)$ is a measured output function.

Let in the LTVSPS (1) some fixed value of the parameter $\mu \in (0, \mu^0]$ was implemented and an unknown initial state

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad x_0 \in \mathbb{R}^{n_1}, \quad y_0 \in \mathbb{R}^{n_2}, \tag{3}$$

which, due to the system (1), (2), generated a process $\{(x(t), y(t)),\ t \in T\}$, inaccessible to direct observation and a error-free measurable output function $v(t) = v(t, \mu, x_0, y_0)$, $t \in T$. It is required to estimate the unknown state $(x(t), y(t))$, $t \in T$ using the known function $v(t)$, $t \in T$. To solve this problem, we will construct an asymptotic observer.

Let's denote $n = n_1 + n_2$, $z' = (x', y')$, $z_0' = (x_0', y_0')$, the symbol "$'$" (prime) indicates transposition.

Using the LTVSPS parameters (1), (2) we define the vector function $c(t) = (c_1(t),\ c_2(t))$, as well as the matrix function depending on the parameter $\mu > 0$

$$A\,(t, \mu) = \begin{pmatrix} A_1(t) & A_2(t) \\ \dfrac{A_3(t)}{\mu} & \dfrac{A_4(t)}{\mu} \end{pmatrix}. \tag{4}$$

Then the system (1)–(3) can be represented in the state space $\mathbb{R}^n$

$$(A_\mu, c): \quad \begin{aligned} &\dot{z}(t) = A\,(t, \mu)\, z(t),\ z \in \mathbb{R}^n,\ t \in T, \\ &v(t) = c\,(t)\, z(t),\ v \in \mathbb{R},\ t \in T, \\ &z(t_0) = z_0. \end{aligned} \tag{5}$$

Let us identify the system (5) with the pair (A_μ, c), consisting of the matrix functions $A(t, \mu)$ and $c(t)$. Let us represent the matrix $A(t, \mu)$ (4) in the form

$$A(t, \mu) = \operatorname{diag}\left\{E_{n_1}, \frac{1}{\mu}E_{n_2}\right\} A(t), \quad A(t) = \begin{pmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{pmatrix}. \tag{6}$$

Here and below, E_k denotes the unit $k \times k$-matrix.

By (6), the system (5), defined by the pair of matrix functions $A(t), c(t)$ and a small parameter $\mu \in (0, \mu^0]$, is also identified with set $\{A, c, \mu\}$. If the parameter μ takes all possible values from the interval $(0, \mu^0]$, then we obtain a μ-parametric family of systems $\{A, c\}_{\mu^0}$, which is considered as a single mathematical object defined on $T \times (0, \mu^0]$. Fixed $\mu \in (0, \mu^0]$ distinguishes a specific system (A_μ, c) from the family $\{A, c\}_{\mu^0}$.

Let ρ be some positive number.

Definition 1. System of differential equations

$$\dot{w}(t) = A(t,\mu)w(t) + K(t,\mu)\left(v(t) - c(t)w(t)\right), \quad w \in \mathbb{R}^n, \tag{7}$$

is called a full-order ρ-exponential observer of the family of systems $\{A,c\}_{\mu^0}$ with a gain vector $K(t,\mu)$, and an estimation coefficient $c_\rho(\mu) > 0$, $\mu \in (0,\mu^0]$, if for any $\bar{t} > t_0$ the estimation error $\varepsilon(t,\mu) = z(t,\mu) - w(t,\mu)$ satisfies the inequality

$$||\varepsilon(t,\mu)|| \leqslant c_\rho(\mu)\exp(-\rho(t-\bar{t})), \forall t \geqslant \bar{t}, \forall \mu \in (0,\mu^0].$$

A method for constructing ρ-exponential observers for uniformly observed time-varying systems with quasi-differentiable coefficients was proposed in [5]. For any fixed μ, this method can be used to construct the LTVSPS observer (5). However, the following problems may arise: in the general case, the observer's parameters will depend on a small parameter that may be unknown in advance; when using the method from [5], the existence of the canonical Frobenius form and the construction of a corresponding parameter dependent transformation matrix for a high-dimensional SPS are required. In this case, for $\mu \to 0$ the transformation matrix in the general case will be ill-conditioned, and the elements of the canonical Frobenius form tend to infinity. Therefore, it is relevant to develop methods for synthesizing observers that are robust with respect to a small parameter, which do not use knowledge of the value of the small parameter and provide "good" estimates of the state for any sufficiently small values.

The conditions for robust P-uniform observability of a linear time-varying two-time-scale LTVSPS, necessary for constructing ρ-exponential observers, were obtained in [31].

We denote by $O(\mu)$ a vector function $f(t,\mu)$ on the interval $[t^1,\infty)$ such that there exist constants $\mu^* > 0$, $d > 0$ such that the Euclidean norm $||f(t,\mu)||$ satisfies the inequality $||f(t,\mu)|| \leqslant d\mu$ $\forall \mu \in (0,\mu^*]$, $\forall t \in [t^1,\infty)$.

Let $r(t,\mu) > 0$ be a given function bounded on $T \times (0,\mu^0]$.

Definition 2. The system (7) is called a full-order ρ-exponential observer with a bounded on $T \times (0,\mu^0]$ error $r(t,\mu)$ for the family of systems $\{A,c\}_{\mu^0}$ if for any $\bar{t} > t_0$ the estimation error satisfies the inequality $||\varepsilon(t,\mu)|| \leqslant c_\rho(\mu)\exp(-\rho(t-\bar{t})) + r(t,\mu)$, $\forall t \geqslant \bar{t}$, $\forall \mu \in (0,\mu^0]$.

If in (7) $K(t,\mu) = \operatorname{diag}\{E_{n_1}, \frac{1}{\mu}E_{n_2}\}K(t)$, $c_\rho(\mu) \equiv c_\rho$, $\mu \in (0,\mu^0]$, and for some n, $n = 1,2,3,\ldots$, the equality $r(t,\mu) = O(\mu^n)$, $t \in T$, is true then we call (7) a robust μ-asymptotic ρ-exponential observer of the LTVSPS family (5).

The definitions introduced above are consistent with the concepts from [5, 14, 16, 34].

Robust μ-asymptotic ρ-exponential observer of the LTVSPS family $\{A,c\}_{\mu^0}$ (5) performs uniform (on μ) asymptotic (on t) estimation of the state vector (x,y) of any system of the LTVSVS family (5) with a bounded error, which has an order of smallness no less than μ. Its gain vector is calculated independently of the small parameter and provides such estimates of the states of systems of the LTVSVS family (5) that the estimation error with an arbitrary predetermined exponential decay rate tends to an infinitesimal value of the order of smallness no less than the small parameter.

Problem 1. To construct for the LTVSPS (1)–(2) a robust μ-asymptotic ρ-exponential observer. In this case, the gain vector must be expressed by gain vectors of parameter-independent subsystems, constructed according to the LTVSPS (1)–(2) and having a dimension smaller than the original one.

3. LTVSPS SUBSYSTEMS, THEIR OBSERVABILITY AND OBSERVERS

3.1. LTVSPS Subsystems and Their Connection with LTVSPS

Solving problems of analysis and synthesis of SPS is often simplified by using the asymptotic decomposition of SPS into subsystems of smaller dimension. This paper describes a constructive method for constructing observers for the LTVSPS, using the asymptotic decomposition of the LTVSPS and implemented through the construction of observers of associated with the LTVSPS (1)–(2) independent of the parameter μ a degenerate system (DS) and a boundary layer systems (BLS) [32], which are obtained from a singularly perturbed system if we consider it separately in the "fast" and "slow" time scales at $\mu = 0$.

Let $\det A_4(t) \neq 0,\ t \in T$, i.e., a standard LTVSPS is being considered. Then the DS (slow subsystem) has the form

$$(A_s, c_s): \quad \begin{array}{l} \dot{x}_s(t) = A_s(t)\, x_s(t),\, x_s(0) = x_0,\ v_s(t) = c_s(t)\, x_s(t),\ t \in T, \\ A_s(t) \triangleq A_1(t) - A_2(t)A_4^{-1}(t)A_3(t),\ c_s(t) \triangleq c_1(t) - c_2(t)A_4^{-1}(t)A_3(t), \end{array} \tag{8}$$

and is a time-varying n_1-dimensional system. Let us identify it with the pair (A_s, c_s).

The BLS (fast subsystem) for the LTVSPS (1)–(2) has the form:

$$(A_4(t_0), c_2(t_0)): \quad \begin{array}{l} \dfrac{dy_f(\tau)}{d\tau} = A_4(t_0)y_f(\tau),\ v_f(\tau) = c_2(t_0)y_f(\tau),\ \tau = \dfrac{t-t_0}{\mu} \in T_\mu \triangleq \left[0, \dfrac{t_1 - t_0}{\mu}\right], \\ y_f(\tau) = y(t_0 + \mu\tau) - A_4^{-1}(t_0)A_3(t_0)x_0,\, y_f(0) = y_0 + A_4^{-1}(t_0)A_3(t_0)x_0, \end{array} \tag{9}$$

and is a linear time-invariant n_2-dimensional system. Let us identify it with the pair $(A_4(t_0), c_2(t_0))$.

Along with the time-invariant BLS (9), we introduce a *t-family of fast subsystems* $(A_4, c_2)(t)$ of the form (9) with $A_4(t), c_2(t)$, (where $t \in T$ is a fixed value, considered as a family parameter) instead of $A_4(t_0), c_2(t_0)$. The BLS (9) is isolated from the t-family of fast subsystems $(A_4, c_2)(t)$ when $t = t_0$.

Note that the DS (A_s, c_s) (8) and the t-family of fast subsystems $(A_4, c_2)(t)$ (9) are defined for the entire family $\{A, c\}_{\mu^0}$ immediately.

The following statement, which follows from [32, Theorem 6.1, p. 227], establishes a connection between the solutions of the LTVSPS (1), (3) and its subsystems (8), (9).

Statement 1. *Let the roots* $\lambda(A_4(t))$ *of the characteristic equation* $\det(\lambda E_{n_2} - A_4(t)) = 0$ *of matrix* $A_4(t)$ *satisfy the inequality* $\operatorname{Re}\lambda(A_4(t)) < -\gamma < 0\ \forall t \in T,\ \gamma = \text{const} > 0$*;* $A_i(t),\ i = \overline{1,4}$ *are continuously differentiable on* T*,* $\dot{A}_k(t), k = \overline{2,4}$*, are bounded on* T*. Then there exists* $\mu^* > 0$ *such that for all* $\mu \in (0, \mu^*]$ *the functions*

$$x^1(t) = x_s(t), \quad y^1(t) = y_f\left(\frac{t-t_0}{\mu}\right) - A_4^{-1}(t)A_3(t)x_s(t),\ t \in T,$$

where $x_s(t), y_f(t)$ *are solutions of the DS* (8) *and the BLS* (9)*, are 1st-order asymptotic approximations of solution of the problem* (1), (3)*, uniform on* $t \in T$ *:*

$$x(t) = x^1(t) + O(\mu),\ y(t) = y^1(t) + O(\mu),\ t \in T.$$

3.2. Observability of Subsystems

In this work, when constructing observers for the DS (8), we use a method that imposes weaker requirements on the smoothness of functions, compared to previously known ones, and uses the concept of quasi-differentiability with respect to some lower triangular matrix P_s, a system of class $\{P_s, n_1 - 1\}$ and uniform observability of the DS. Let us introduce the concepts related to this.

For an arbitrary non-negative integer k, we denote by $\mathcal{U}_k(T)$ the set of all lower triangular matrices $P(t)$ of dimension $((k+1)\times(k+1))$ with elements $p_{ji}(t)$ continuous on T $(j,i=0,1,\dots,k)$ satisfying the condition $p_{jj}(t)\neq 0$ $(t\in T)$, $(j=0,1,\dots,k)$. For an arbitrary matrix $P(t)$ from the set $\mathcal{U}_k(T)$ and a continuous function $w:T\to\mathbb{R}$ quasi-derivatives ${}^{j}_{P}w(t)$ of order j $(j=0,1,\dots,k)$ with respect to the matrix $P(t)$ are determined according to recurrent rules [23]:

$$\begin{gathered}{}^{0}_{P}w(t)=p_{00}(t)w(t),\quad {}^{1}_{P}w(t)=p_{11}(t)\frac{d({}^{0}_{P}w(t))}{dt}+p_{10}(t)({}^{0}_{P}w(t)),\dots,\\ {}^{j}_{P}w(t)=p_{jj}(t)\frac{d({}^{j-1}_{P}w(t))}{dt}+\sum_{i=0}^{j-1}p_{ji}(t)({}^{i}_{P}w(t))\quad (j=2,3,\dots,k).\end{gathered}\tag{10}$$

It is assumed that the differentiation operations in the formulas (10) are feasible and lead to continuous functions.

Let some matrix $P_s\in\mathcal{U}_{n_1}(T)$ be given. Following [24, page. 31], we introduce the definition for the DS (8).

Definition 3. The DS (A_s,c_s) (8) has a P_s-class m if each of its output functions $v_s(t)=v_s(t,x_0)$, $t\in T$, has continuous quasi-derivatives with respect to the matrix P_s up to order m inclusive.

Applying Lemma 2.1 from [24, p. 32] to the system (8) we obtain

Statement 2. *The DS* (8) *has the* P_s*-class* n_1-1 *if and only if for any* $k=1,\dots,n_1-1$ *following row vectors exist and are continuous*

$$s_{s0}(t)=p_{s,00}(t)c_s(t),\ \ s_{sj}(t)=p_{s,jj}(t)\big(s_{s,j-1}(t)A_s(t)+\dot{s}_{s,j-1}(t)\big)+\sum_{i=0}^{j-1}p_{s,ji}(t)s_{si}(t).\tag{11}$$

In particular, the DS (8) has the class $\{P_s,n_1-1\}$ with respect to the $n_1\times n_1$-matrix P_s of the form (18) from [5], constructed according to the parameters of the DS (8).

Definition 4. The DS (A_s,c_s) (8) of class $\{P_s,n_1-1\}$ is called P_s*-uniformly observable on the interval* T if for any $x_0\in\mathbb{R}^{n_1}$ mapping

$$x_s(t)\to\big({}^{0}_{P_s}v_s(t),\,{}^{1}_{P_s}v_s(t),\dots,\,{}^{n_1-1}_{P_s}v_s(t)\big),\ v_s(t)=v_s(t,x_0)$$

is injective for each $t\in T$.

Definition 5. The t-family of fast subsystems $(A_4,c_2)(t)$ (9) is called completely observable on T_μ, if any subsystem from the t-family $(t\in T)$ is completely observable [35, p. 68; 24, p. 29].

Let the DS (A_s,c_s) (8) have P_s-class n_1-1. Let us define the $(n_1\times n_1)$ observability matrix of the DS (A_s,c_s):

$$S_{P_s}(t)=\begin{pmatrix}s_{s0}(t)\\ s_{s1}(t)\\ \dots\dots\\ s_{s,n_1-1}(t)\end{pmatrix},\quad (t\in T),$$

where n_1-row vectors $s_{sj}(t)$, $j=1,2,\dots$ are determined by the formulas (11) and $(n_2\times n_2)$-observability matrix of t-family of fast subsystems $(A_4,c_2)(t)$:

$$S_f(t)=\begin{pmatrix}s_{f0}(t)\\ s_{f1}(t)\\ \dots\dots\\ s_{f,n_2-1}(t)\end{pmatrix},\quad (t\in T),\tag{12}$$

where n_2-row vectors $s_{f0}(t),\ s_{f1}(t),\ \dots$ are defined by the formulas

$$s_{fj}(t)=s_{f,j-1}(t)A_4(t),\quad s_{f0}(t)=c_2(t).\tag{13}$$

Applying the conditions from [35, p. 68; 36, p. 89] to the DS (8) and to the t-family of fast subsystems (9), we obtain

Statement 3. *The DS* (8) *of P_s-class $n_1 - 1$ is P_s-uniformly observable on T if and only if* rank $S_{P_s}(t) = n_1 \quad \forall\, t \in T$.

Statement 4. *The t-family of fast subsystems $(A_4, c_2)(t)$* (9) *is completely observable if and only if* rank $S_f(t) = n_2 \quad \forall\, t \in T$.

3.3. Observers for Subsystems

Let ρ_s, ρ_f be some positive numbers.

Definition 6. A system of differential equations

$$\dot{w}_s(t) = A_s(t)w_s(t) + k_s(t)\left(v_s(t) - c_s(t)w_s(t)\right), \quad w_s \in \mathbb{R}^{n_1}, \ \ t > t_0, \tag{14}$$

is called a ρ_s-exponential observer of the DS (A_s, c_s) (8) with the gain vector $k_s(t)$ and the estimation coefficient $c_{\rho s} > 0$, if the estimation error $\varepsilon_s(t) = x_s(t) - w_s(t)$ satisfies the inequality $||\varepsilon_s(t)|| \leqslant c_{\rho s} \exp(-\rho_s(t - \bar{t}))$, $t \geqslant \bar{t}$, for any $\bar{t} > t_0$.

Definition 7. A system of differential equations

$$\frac{dw_f(\tau)}{d\tau} = A_4(t)w_f(\tau) + k_f(t)\left(v_f(\tau) - c_2(t))w_f(\tau)\right), \quad w_f \in \mathrm{R}^{n_2}, \ \ \tau > 0, \quad t \in T, \tag{15}$$

is called a ρ_f-exponential observer of the t-family of fast subsystems $(A_4, c_2)(t)$ with the gain vector $k_f(t)$ and the estimation coefficient $c_{\rho f} > 0$, if for any system of the t-family $(\forall t \in T)$ the error $\varepsilon_f(\tau) = y_f(\tau) - w_f(\tau)$ satisfies the inequality $||\varepsilon_f(\tau)|| \leqslant c_{\rho f} \exp\left(-\rho_f(\tau - \bar{\tau})\right)$, $\tau \geqslant \bar{\tau}$, for any $\bar{\tau} > 0$.

We denote by $\mathcal{L}(n_1)$ the set of all invertible $n_1 \times n_1$ matrices that are continuously differentiable and bounded on T together with their inverses. The set $\mathcal{L}(n_1)$ is a Lyapunov group [37]. The action of the group $\mathcal{L}(n_1)$ on the pair (A, c), consisting of a $n_1 \times n_1$-matrix function and a n_1-row vector with elements continuous on T, we define according to the following rule [24, p. 42]

$$G * (A, c) = \left(G^{-1}AG - G^{-1}\dot{G},\, cG\right), \quad G \in \mathcal{L}(n_1). \tag{16}$$

Using [5, 31], it is easy to verify the validity of the following statements.

Statement 5. *If for some $P_s \in \mathcal{U}_{n_1}(T)$ the DS* (8) *is P_s-uniformly observable and for it there exists a canonical Frobenius form (A_s^0, c_s^0),*

$$A_s^0\,(t) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \alpha_0 \\ 1 & 0 & 0 & \dots & 0 & \alpha_1 \\ 0 & 1 & 0 & \dots & 0 & \alpha_2 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 1 & \alpha_{n_1-1} \end{pmatrix}, \quad c_s^0\,(t) = (0, 0, \dots, 0, 1)', \tag{17}$$

relative to the actions of the Lyapunov group $\mathcal{L}(n_1)$, then for any $\rho_s > 0$ there exists a ρ_s-exponential observer (14).

Statement 6. *If the t-family of fast subsystems* (9) *is completely observable, then for any $\rho_f > 0$ there exists a ρ_f-exponential observer* (15).

Observers for the DS (8) and the t-family (9) can be constructed by applying for them the method of constructing ρ-exponential observers from [5, Theorem 5].

3.3.1. Scheme for Constructing a ρ-Exponential Observer for a Time-Varying DS (8).

1) Let the DS (8) be uniformly observable (Statement 3) and for it there exists a canonical Frobenius form (A_s^0, c_s^0), with respect to the actions of the Lyapunov group $\mathcal{L}(n_1)$ [24, p. 69]. Let us denote by $\alpha_s(t) = (\alpha_0, \alpha_1, \ldots, \alpha_{n_1-1})'$ n_1-vector column of coefficients of the matrix $A_s^0(t)$.

2) Find the transformation $G_s(t) \in \mathcal{L}(n_1)$ for which $G_s * (A_s, c_s) = (A_s^0, c_s^0)$.

Note that to construct the canonical Frobenius form and the corresponding transformation matrix $G_s(t)$ from the Lyapunov group $\mathcal{L}(n_1)$ you can use the method described in [5; 24, p. 81].

3) Choose real numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$, satisfying for the given $\rho > 0$ the inequality $\lambda_i < -\rho$, $i = 1, 2, \ldots, n_1$.

4) Construct a polynomial $(\xi - \lambda_1)(\xi - \lambda_2) \cdots (\xi - \lambda_{n_1}) = \xi^{n_1} - \beta_{n_1-1}\xi^{n-1} - \cdots - \beta_1\xi - \beta_0$ and let $\beta_s = (\beta_0, \beta_1, \ldots, \beta_{n_1-1})'$ be n_1-vector column..

5) Calculate the gain vector $k_s(t)$ for the observer of DS (14):

$$k_s(t) = G_s(t)(\alpha_s(t) - \beta_s). \tag{18}$$

3.3.2. Scheme for Constructing a ρ-Exponential Observer for a t-Family of Time-Invariant Fast Subsystems.

1) Let the t-family of fast subsystems (9) be completely observable (Statement 4). By the canonical Frobenius form (A_f^0, c_f^0) of the t-family (9) we mean a set of systems, each of which is the canonical Frobenius form of the corresponding fast subsystem. To find it, we construct a characteristic polynomial of the t-family (9). For each $t \in T$ its coefficients define the vector $\alpha_f(t) = (\alpha_0(t), \alpha_1(t), \ldots, \alpha_{n_2-1}(t))'$ coefficients of the canonical Frobenius form of the corresponding fast subsystem.

2) Calculate the transition matrix to the canonical Frobenius form for the t-family of fast subsystems: $G_f(t) = S_f^{-1}(t)S_f^0(t)$, where $S_f(t)$ and $S_f^0(t)$ are the observability matrices of the t-family of fast subsystems and its canonical Frobenius form, calculated using the formulas (12), (13) with matrices $A_4(t), c_2(t)$ and $A_f^0(t), c_f^0(t)$, respectively.

3) Choose real numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n_2}$ that, for a given positive number ρ, satisfy the inequality $\lambda_i < -\rho,\ i = 1, 2, \ldots, n_2$.

4) Form a n_2-vector $\beta_f = (\beta_0, \beta_1, \ldots, \beta_{n_2-1})'$ with constant elements such that $(\xi - \lambda_1)(\xi - \lambda_2) \cdots (\xi - \lambda_{n_2}) = \xi^{n_1} - \beta_{n_1-1}\xi^{n-2} - \cdots - \beta_1\xi - \beta_0$.

5) Calculate the gain vector $k_f(t)$ for the observer of the t-family of fast subsystems:

$$k_f(t) = G_f(t)\left(\alpha_f(t) - \beta_f\right). \tag{19}$$

Note that the function $k_f(t)$, $t \in T$, defined in this way inherits the properties of smoothness and continuity of the functions $\alpha_f(t)$, and hence the coefficients $A_4(t), c_2(t)$ of the t-family of fast subsystems.

4. COMPOSITE OBSERVER OF LTVSVS

4.1. Robust μ-Asymptotic ρ-Exponential Observer of LTVSPS

Let $\rho > 0$ be given.

Theorem 1. *Let*

(i) *for some matrix $P_s \in \mathcal{U}_{n_1}(T)$ the conditions of Statements 2, 3 are satisfied and DS (A_s, c_s) (8) has the canonical Frobenius form under the action of the Lyapunov group $\mathcal{L}(n_1)$;*

(ii) *for the DS (8) a ρ-exponential observer with a gain vector $k_s(t)$ (18) and an estimation coefficient c_{ρ_s} was constructed;*

(iii) *the conditions of the Statement 4 are satisfied;*

(iv) *for the t-family of fast subsystems* $(A_4, c_2)(t)$ (9) *a* $\mu^0\rho$*-exponential observer with a gain vector* $k_f(t)$ (19) *and an estimation coefficient* c_{ρ_f} *was constructed;*

(v) *matrix functions*

$$\begin{aligned} \tilde{A}_1(t) &= A_1(t) - k_1(t)c_1(t), \quad \tilde{A}_2(t) = A_2(t) - k_1(t)c_2(t), \\ \tilde{A}_3(t) &= A_3(t) - k_2(t)c_1(t), \quad \tilde{A}_4(t) = A_4(t) - k_2(t)c_2(t), \end{aligned} \tag{20}$$

where

$$k_1(t) = A_2(t)A_4^{-1}(t)k_f(t) + k_s(t)[E_{n_2} - c_2(t)A_4^{-1}(t)k_f(t)], \quad k_2(t) = k_f(t), \tag{21}$$

are continuously differentiable and bounded, derivatives of functions $\tilde{A}_i(t)$, $i = 2, 3, 4$, *are bounded on* T*;*

(vi) $\operatorname{Re}\lambda(\tilde{A}_4(t)) \leqslant -\gamma_1 < 0$, $\gamma_1 = \text{const} > 0$, $\forall t \geqslant t_0$.

Then there exists $\mu^* \in (0, \mu^0]$ *such that the system*

$$\begin{gathered} \dot{w}_x(t) = \tilde{A}_1(t)w_x(t) + \tilde{A}_2(t)w_y(t) + k_1(t)v(t), \quad w_x \in \mathbb{R}^{n_1}, \quad w_y \in \mathbb{R}^{n_2}, \\ \mu\dot{w}_y(t) = \tilde{A}_3(t)w_x(t) + \tilde{A}_4(t)w_y(t) + k_2(t)v(t), \quad t > t_0, \\ w_x(0) = 0, \quad w_y(0) = 0, \end{gathered} \tag{22}$$

is a robust μ*-asymptotic* ρ*-exponential observer of the LTVSPS* (5) *family* $\{A, c\}_{\mu^*}$.

The proof of Theorem 1 is given in Appendix.

Note that the requirement for the existence of a canonical Frobenius form for the DS weakens the requirement for the existence of a canonical Frobenius form for the original LTVSPS. For example, a system like $\dot{x}(t) = |t-1|x(t) + (1 - |t-1|)y(t)$, $\mu\dot{y}(t) = tx(t) - ty(t)$, $v(t) = y(t)$ does not have the canonical Frobenius form [24, p. 73, Theorem 3.4]. At the same time, the DS $\dot{x}_s(t) = x_s(t)$, $v_s(t) = x_s(t)$, for this system has the form of the canonical Frobenius form. For a t-family of time-invariant fast subsystems, the canonical Frobenius form always exists if the condition of complete observability is satisfied (Statement 4). For the example given here, the DS has the form $\frac{d}{d\tau}y_f(\tau) = -t\,y_f(\tau), v_f(\tau) = y_f(\tau)$ and is completely observable.

4.2. Algorithm for Constructing a Robust μ*-Asymptotic* ρ*-Exponential Observer of the LTVSPS* (5)

The following algorithm follows from the proof of Theorem 1 (see Appendix).

1. Construct the DS (8) and check the fulfillment of the conditions (i) of Theorem 1. If they are not satisfied for any P_s, then the DS is not uniformly observable and, therefore, there is no canonical Frobenius form for it. Notice, that one of the matrices P_s for which the conditions of the Statement 2 are satisfied is a matrix P_s of the form (18) from [5], which is constructed according to the parameters of the DS (8).

2. Construct the t-family of fast subsystems (9) and check the fulfillment of the conditions (iii) of Theorem 1.

3. Set the desired value for the rate of exponential decrease in observation errors $\rho > 0$.

4. Calculate the gain vector $k_s(t)$ of ρ-exponential observer for the DS (8) according to (18) (item 3.3.1).

5. Calculate the gain vector $k_f(t)$ of $\mu^0\rho$-exponential observer for the t-family of fast subsystems (9) using (19) (item 3.3.2).

6. Calculate the coefficients $k_1(t)$, $k_2(t)$ using (21).

7. Check the fulfillment of the conditions (v) and (vi) of Theorem 1.

8. Form composite observer (22).

5. REDUCED OBSERVERS OF LTVSPS

The observer's state (22) approaches the $O(\mu)$-approximation of the LTVSPS state (5) with an exponential rate ρ, which can be chosen arbitrarily. However, if the value of the small parameter μ is very small or unknown, then it is difficult to practically implement the observer (22). In this regard, it is advisable to evaluate the state of the original system using a system that does not have "fast" observer modes (22).

Similar to [16], we introduce two reduced observers of the LTVSPS.

According to the first approach, an asymptotic Luenberger observer (14) is constructed for the DS (8) of the original LTVSPS (5). In [16] for time-invariant SPS it is proven that if matrices of the DS (8) and the BLS (9) are Hurwitz and the DS is observable, then the asymptotic observer of the DS is a μ-asymptotic observer of the original LTISPS. Following the scheme of proof of Theorem 4 from [16] using Theorem 6.1. [32, p. 227], a similar result can be proven for the LTVSPS.

Theorem 2. *Let the conditions of Statements 1, 5 be satisfied. Then there exists $\mu^* > 0$ such that the system*

$$\begin{gathered}\dot{w}_{sx}(t) = (A_s(t) - k_s(t)c_s(t))\, w_{sx}(t) + k_s(t)v(t),\\ w_{sy} = -A_4^{-1}(t)A_3(t)w_{sx}(t), \quad t > t_0,\\ w_{sx} \in \mathbb{R}^{n_1}, \quad w_{sy} \in \mathbb{R}^{n_2}, \quad w_{sx}(0) = 0,\end{gathered} \tag{23}$$

is a robust μ-asymptotic ρ-exponential observer of the LTVSPS (5) *family* $\{A, c\}_{\mu^*}$.

According to the second approach, the degenerate system is constructed for the observer (22), which is taken to be the observer of the original LTVSPS (5).

Theorem 3. *Let the conditions of Theorem 1 be satisfied. Then there exists $\mu^* > 0$ such that the system*

$$\begin{gathered}\dot{w}_{xs}(t) = \left(\tilde{A}_1(t) - \tilde{A}_2(t)\tilde{A}_4^{-1}(t)\tilde{A}_3(t)\right) w_{xs}(t) + \left(k_1(t) - \tilde{A}_2(t)\tilde{A}_4^{-1}(t)k_2(t)\right) v(t),\\ w_{ys}(t) = -\tilde{A}_4^{-1}(t)\tilde{A}_3(t)w_{xs}(t) - \tilde{A}_4^{-1}(t)k_2(t)v(t), \quad t > t_0,\\ w_{xs} \in \mathbb{R}^{n_1}, \quad w_{ys} \in \mathbb{R}^{n_2}, \quad w_{xs}(0) = 0,\end{gathered} \tag{24}$$

is a robust μ-asymptotic ρ-exponential observer of the LTVSPS (5) *family* $\{A, c\}_{\mu^*}$.

6. EXAMPLES

Let us consider numerical examples illustrating the application of the proposed method for constructing robust μ-asymptotic ρ-exponencial LTVSPS observers. The practical implementation of the method uses the algorithm for constructing a robust μ-asymptotic ρ-exponential observer of the LTVSPS (5) outlined in Section 4.2 (items 1–8) and schemes for constructing ρ-exponential observers for subsystems from sections 3.3.1 and 3.3.2.

Example 1. Let's consider LTVSPS

$$\begin{gathered}\dot{x}_1(t) = (\alpha(t) - 1)x_2(t) + (2 - \alpha(t))y(t), \quad \dot{x}_2(t) = -x_1(t) - x_2(t),\\ \mu\dot{y}(t) = x_2(t) - y(t), \quad v(t) = y(t), \quad t \in T,\end{gathered} \tag{25}$$

whose matrices have the form: $A_1(t) = \begin{pmatrix} 0 & \alpha(t) - 1 \\ -1 & -1 \end{pmatrix}$, $A_2(t) = \begin{pmatrix} 2 - \alpha(t) \\ 0 \end{pmatrix}$, $A_3(t) = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $A_4(t) = \begin{pmatrix} -1 \end{pmatrix}$, $c_1(t) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $c_2(t) = \begin{pmatrix} 1 \end{pmatrix}$ and the function $\alpha(t)$ is bounded and continuously differentiable on T with bounded derivative.

1. The degenerate system for the LTVSPS (25):

$$\dot{x}_{s1}(t) = x_{s2}(t), \quad \dot{x}_{s2}(t) = -x_{s1}(t) - x_{s2}(t), \quad v_s(t) = x_{s2}(t), \tag{26}$$

where $A_s = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $c_s = \begin{pmatrix} 0, & 1 \end{pmatrix}$, is time-invariant and has a non-singular observability matrix $S_s(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, this means that for the DS (26) there is a canonical Frobenius form and, according to Statement 3, the condition (i) of Theorem 1 is satisfied.

The transformation (16) of the DS (26) by using matrix $G_s(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ leads to the canonical Frobenius form (A_s^0, c_s^0), $A_s^0(t) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $c_s^0(t) = (0,\, 1)$, where $\alpha_s(t) = (-1,\, -1)'$.

2. The t-family of fast subsystems for LTVSPS (25)

$$\frac{d\tilde{y}(\tau)}{d\tau} = -\tilde{y}(\tau), \quad \tilde{v}_f(\tau) = \tilde{y}(\tau), \quad t \in T, \tag{27}$$

has an observability matrix $S_f(t) = (1)$, rank $S_f(t) = 1$, $\forall t \in T$, then, according to Statement 4, the condition (iii) of Theorem 1 is satisfied.

3. Let us set the rate of exponential decrease in observer errors: $\rho = 2$.

4. Let's take $\lambda_i = -3, i = 1, 2$ and calculate $\beta_s = (-9, -6)'$ and $k_s(t) = (-8,\, 5)'$.

5. The t-family of fast subsystems (27) already has a Frobenius form with $\alpha_f = (-1)$, so $G_f(t) = 1$. Let's choose $\lambda_f = -3$, then $\beta_f = (-3)$ and $k_f(t) =(2)$.

6. Calculate the coefficients (21): $k_1(t) = (2\alpha(t) - 28,\, 15)'\,, k_2 = (2)$.

7. Matrix functions (20) $\tilde{A}_1(t) = A_1(t)$, $\tilde{A}_2(t) = (-3\alpha(t) + 30,\, -15)'$, $\tilde{A}_3(t) = (0, 1)$, $\tilde{A}_4(t) = (-3)$ for LTVSPS (25) satisfy the conditions (v), (vi) of Theorem 1.

8. Finally, the robust μ-asymptotic ρ-exponential composite full-order observer (22) for the LTVSPS (25) for $\rho = 2$ will take the form:

$$\begin{aligned} \dot{w}_{x1}(t) &= (\alpha(t) - 1)w_{x2}(t) - 3(\alpha(t) - 10)w_y(t) + 2(\alpha(t) - 14)v(t), \\ \dot{w}_{x2}(t) &= -w_{x1}(t) - w_{x2}(t) - 15w_y(t) + 15v(t), \\ \mu\dot{w}_y(t) &= w_{x2}(t) - 3w_y(t) + 2v(t). \end{aligned} \tag{28}$$

Figures 1, 2 and Table 1 show the results of numerical experiments (performed using Wolfram Mathematica) with the model (25) with initial conditions $x_1(0) = 1$, $x_2(0) = 0$, $y(0) = 0$, with $\alpha(t) = \sin(t)$,

Figure 1 shows the dynamics of errors ε_{x1} (thick), ε_{x2} (thin), ε_y (dashed line) of the composite observer (22) for the LTVSPS (25) at $\mu = 0.01$. Figure 1a corresponds to the choice of $\lambda_i = -3$, Fig. 1b corresponds to $\lambda_i = -6$ and demonstrates the change in the dynamics of observer errors with increasing exponential decay rate.

To compare the quality of estimation for different values of the small parameter, we calculate the integral norm of the observer (28) errors on the interval $[0, 30]$ (Table 1).

Table 1. Integral norm of the observer (28) errors, $\alpha(t) = \sin(t)$

	$\mu = 0.5$	$\mu = 0.1$	$\mu = 0.01$
$\|\|\varepsilon_{x1}\|\|_1$	0.6755937	0.668459	0.666458
$\|\|\varepsilon_{x2}\|\|_1$	0.112656	0.111410	0.111141
$\|\|\varepsilon_y\|\|_1$	0.037552	0.0371366	0.030469

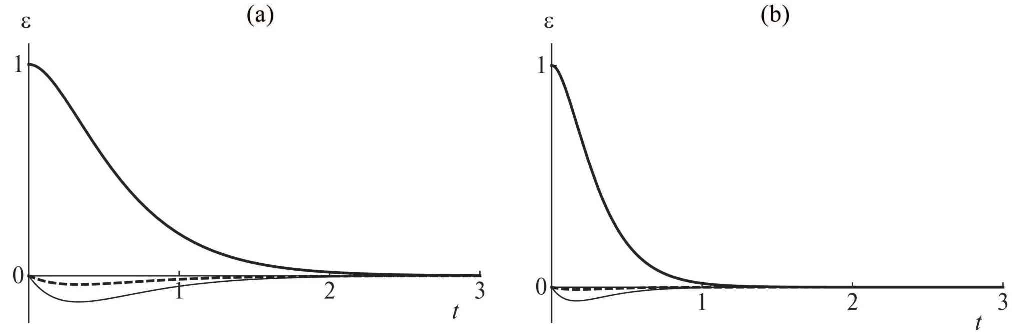

Fig. 1. Dynamics of errors of the composite observer (22) for the LTVSPS (25).

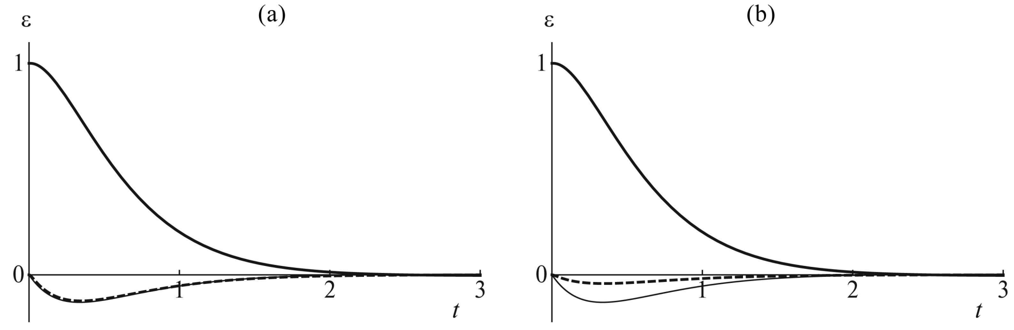

Fig. 2. Dynamics of errors of reduced observers (29) and (30) for the LTVSPS (25).

Comparison of error estimates from Table 1 confirms that as μ decreases, the estimation error decreases.

The reduced observer (23) for the LTVSPS (25) has the form

$$\begin{gathered}\dot w_{sx1}(t) = 9w_{sx2}(t) - 8v(t),\\ \dot w_{sx2}(t) = -w_{sx1}(t) - 6w_{sx2}(t) + 5v(t),\\ w_{sy} = w_{sx2}(t), \quad w_{sx1}(0) = 0, \quad w_{sx2}(0) = 0.\end{gathered} \tag{29}$$

The reduced observer (24) for the LTVSPS (25) has the form

$$\begin{gathered}\dot w_{xs1}(t) = 9w_{xs2}(t) - 8v(t), \quad w_{xs1}(0) = 0,\\ \dot w_{xs2}(t) = -w_{xs1}(t) - 6w_{xs2}(t) + 5v(t), \quad w_{xs2}(0) = 0,\\ w_{ys}(t) = \frac{1}{3}w_{xs2}(t) + \frac{2}{3}v(t).\end{gathered} \tag{30}$$

Error dynamics ε_{x1} (thick), ε_{x2} (thin), ε_y (dashed line) of reduced observers (29) and (30) for the LTVSPS (25) at $\alpha(t) = \sin(t)$, $\mu = 0.01$, $\lambda_i = -3$ shown in Fig. 2: Fig. 2a for observer (29) and Fig. 2b for observer (30).

Example 2. Let's consider LTVSPS

$$\begin{gathered}\dot x_1(t) = \left(\varphi(t) - \frac{\dot\gamma(t)}{\gamma(t)} - \delta(t)\right)x_1(t) + \zeta(t)x_2(t) + \delta(t)y(t),\\ \dot x_2(t) = (\gamma(t) - \alpha(t))\,x_1(t) + \xi(t)x_2(t) + \alpha(t)y(t),\\ \mu\dot y(t) = x_1(t) - y(t),\\ v(t) = -x_1(t) + x_2(t) + y(t),\ t \in T,\end{gathered} \tag{31}$$

where $\gamma(t) = \sin(t) + 2,\ \varphi(t) = \sin(t) + 1,\ \zeta = \cos(t),\ \xi = -\sin(t) - 1,\ \delta(t) = \sin(t) + 1 - \frac{\cos(t)}{\sin(t)+2}$, $\alpha(t)$ is not twice continuously differentiable at at least one point $t \in T$.

The system (31) in the form (1)–(2) has parameters $n_1 = 2,\ n_2 = m = 1$ and matrices:

$$A_1 = \begin{pmatrix} \varphi(t) - \dfrac{\dot\gamma(t)}{\gamma(t)} - \delta(t) & \zeta(t) \\ \gamma(t) - \alpha(t) & \xi(t) \end{pmatrix}, \quad A_2 = \begin{pmatrix} \delta(t) \\ \alpha(t) \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -1 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 \end{pmatrix}.$$

For LTVSPS (31) the matrix function $A(t,\mu)$ (4) is not twice continuously differentiable and for such a system there is no classical observability matrix. Therefore, constructing an observer according to the scheme [5] directly for the LTVSPS (31) is impossible. However, as shown below, it is possible to construct an observer (22) for the system (31).

1. The DS (8) for the LTVSPS (31)

$$\begin{aligned} \dot{\bar x}_1(t) &= \left(\varphi(t) - \frac{\dot\gamma(t)}{\gamma(t)}\right)\bar x_1(t) + \zeta(t)\bar x_2(t), \\ \dot{\bar x}_2(t) &= \gamma(t)\bar x_1(t) + \xi(t)\bar x_2(t), \\ \bar v_s(t) &= \bar x_2(t), \quad t \in T, \end{aligned} \tag{32}$$

has the class $\{E_2, 2\}$. For the DS (32) the classical observability matrix is defined: $S_{E_2}(t) = \begin{pmatrix} 0 & 1 \\ \gamma & \xi \end{pmatrix}$. Since $rank\, S_{E_2}(t) = 2 = n_1,\ \forall t \in T$, then according to the Statement 3 the DS (32) is uniformly observable and the condition (i) of Theorem 1 is satisfied.

2. The t-family of fast subsystems for the LTVSPS (31) coincides with (27), which means that condition (iii) of Theorem 1 is satisfied.

3. Let us set the rate of exponential decrease in observer errors: $\rho = 2$.

4. Choose $\lambda_i = -3$ and calculate $\beta_s = (-9, -6)'$.

5. The transformation (16) of the DS (32) using matrix $G_s(t) = \begin{pmatrix} \frac{1}{\gamma} & \frac{\varphi}{\gamma} \\ 0 & 1 \end{pmatrix}$ leads to the canonical Frobenius form (17) with $\alpha_s(t) = (-\dot\varphi - \xi\varphi + \zeta\gamma,\ \varphi + \xi)'$. Calculated gain vectors for subsystems: $k_s(t) = \begin{pmatrix} \gamma^{-1}\left(9 + 6\varphi + \varphi^2 - \dot\phi\right) + \zeta \\ 6 + \xi + \varphi \end{pmatrix}$, $k_f = (2)$.

6. By (21) we have: $k_1(t) = \begin{pmatrix} -2\delta + 3\zeta + 3\gamma^{-1}\left((\varphi+3)^2 - \dot\varphi\right) \\ -2\alpha + 3(6 + \xi + \varphi) \end{pmatrix}$, $k_2(t) = (2)$.

7. Matrix functions (20)

$$\tilde A_1(t) = \begin{pmatrix} 3(\zeta - \delta) + \varphi + \gamma^{-1}\left(3(\varphi+3)^2 - \dot\gamma - 3\dot\varphi\right) & 2(\delta - \zeta) - 3\gamma^{-1}\left((\varphi+3)^2 - \dot\varphi\right) \\ -3\alpha + \gamma + 3(6 + \xi + \varphi) & 2\alpha + \xi - 3(6 + \xi + \varphi) \end{pmatrix},$$

$$\tilde A_2(t) = \begin{pmatrix} 3\delta - 3\zeta - 3\gamma^{-1}\left((\varphi+3)^2 - \dot\varphi\right) \\ 3\alpha - 3(6 + \xi + \varphi) \end{pmatrix}, \tilde A_3(t) = \begin{pmatrix} 3 & -2 \end{pmatrix}, \tilde A_4(t) = \begin{pmatrix} -3 \end{pmatrix}$$

for LTVSPS (31) satisfy the conditions *(v), (vi)* of Theorem 1.

8. According to Theorem 1, the robust μ-asymptotic ρ-exponential composite full-order observer for LTVSPS (31) for $\rho = 2$ has the form (22) with the coefficients $k_1(t)$, $k_2(t)$ calculated in item 6.

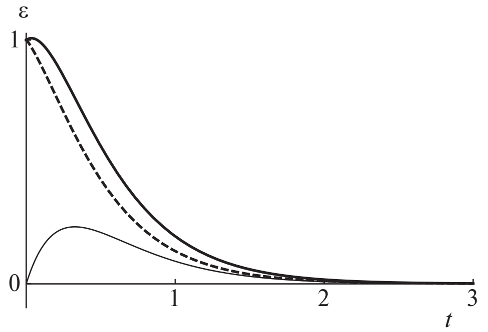

Fig. 3. Dynamics of composite observer (22) errors for the LTVSPS (31).

Figure 3 shows the dynamics of errors ε_{x1} (thick), ε_{x2} (thin), ε_y (dashed line) of composite observer (22) for the LTVSPS (31) with initial conditions $x_1(0) = 1$, $x_2(0) = 0$, $y(0) = 1$ at $\mu = 0.01$.

7. CONCLUSION

The method proposed in this work for synthesizing observers of the LTVSPS states allows us to split the problem into solving independent subproblems of synthesizing observers for systems of smaller dimension, some of which are time-invariant, to ensure the robustness of observers with respect to the small parameter and to significantly weaken the known requirements for the smoothness of coefficients. The vector of gains of the composite observer is expressed through the gains of subsystems independent of the small parameter, corresponding to the separation of time scales. The state estimation error with an arbitrary predetermined exponential decay rate converges to an infinitesimal value of the same order of smallness as the small parameter.

Theorem 1 provides sufficient conditions for the existence of a robust μ-asymptotic ρ-exponential LTVSPS observer. The μ-asymptotic composite full-order (22) and reduced-order (23), (24) observers are constructed. A constructive algorithm for calculating the gain vector (18), (19), (21) of the composite observer is presented, and illustrative examples are given.

When constructing the robust μ-asymptotic ρ-exponential observer for the LTVSPS ρ should be chosen so as to ensure the desired rate of convergence of observation errors into the $O(\mu)$-neighborhood of zero.

Note that when constructing the μ-asymptotic ρ-exponential LTVSPS observer, the existence of a canonical Frobenius form and quasi-differentiability of the output functions of the original LTVSPS are not required.

The results obtained can be used in the design of control systems, identification and diagnostics of dynamic systems described by linear time-varying singularly perturbed systems.

FUNDING

This work was supported in part by the Ministry of Education of the Republic of Belarus, State Program of Scientific Research of the Republic of Belarus for 2021–2025 (project "Convergence 1.2.04").

APPENDIX

Proof of Theorem 1.

It follows from (i) that the DS (A_s, c_s) (8) is uniformly observable and for it there exists a ρ-exponential observer. Under assumption (iii), there exists a $\mu^0\rho$-exponential observer for the t-family of fast subsystems (9).

We will look for a full-order observer for the LTVSPS (5) in the form of the system (7). We will look for the gain vector $K(t,\mu)$ in the form

$$K(t,\mu) = \operatorname{diag}\left\{E_{n_1}, \frac{1}{\mu}E_{n_2}\right\}\begin{pmatrix} k_1(t) \\ k_2(t) \end{pmatrix}, \quad k_1(t) \in \mathbb{R}^{n_1}, \quad k_2(t) \in \mathbb{R}^{n_2},$$

where $k_1(t), k_2(t)$ are not yet defined. Then the observer (7) will take the form (22), and the error dynamics equations $\varepsilon_x(t,\mu) = x(t,\mu) - w_x(t,\mu)$, $\varepsilon_y(t,\mu) = y(t,\mu) - w_y(t,\mu)$ for the observer (22) will look like:

$$\begin{aligned} \dot{\varepsilon}_x(t) &= \tilde{A}_1(t)\varepsilon_x(t) + \tilde{A}_2(t)\varepsilon_y(t), \quad \varepsilon_x \in \mathbb{R}^{n_1}, \\ \mu\dot{\varepsilon}_y(t) &= \tilde{A}_3(t)\varepsilon_x(t) + \tilde{A}_4(t)\varepsilon_y(t), \quad \varepsilon_y \in \mathbb{R}^{n_2}, \quad t > t_0. \end{aligned} \tag{A.1}$$

Since the observation error dynamics system (A.1) has the form of the LTVSPS (1)–(2), then when the assumptions (v), (vi) of Theorem 1 for the error system (A.1) the conditions of Theorem 3.1 from [32, p. 212] and, therefore, there is a decoupling Lyapunov transformation of the form (3.4) from [32, p. 210] with continuously differentiable matrices $\tilde{L}(t,\mu), \tilde{H}(t,\mu)$ bounded on T, which satisfy the following system (in order not to clutter the notation, we will omit the dependence of functions on the argument t in some places):

$$\tilde{A}_3 - \tilde{A}_4\tilde{L}(\mu) + \mu\tilde{L}(\mu)\left(\tilde{A}_1 - \tilde{A}_2\tilde{L}(\mu)\right) = \mu\dot{\tilde{L}}(\mu), \tag{A.2}$$

$$\mu\left[\tilde{A}_1 - \tilde{A}_2\tilde{L}(\mu)\right]\tilde{H}(\mu) - \tilde{H}(\mu)\left[\tilde{A}_4 + \mu\tilde{L}(\mu)\tilde{A}_2\right] + \tilde{A}_2 = \mu\dot{\tilde{H}}(\mu). \tag{A.3}$$

From (A.2), (A.3) taking into account (i), (vi) we have the approximation:

$$\begin{aligned} \tilde{L}(t,\mu) &= A_4^{-1}(t)\,k_2(t)c_2(t)\tilde{L}(t,\mu)\tilde{A}_3(t) + O(\mu), \\ \tilde{L}(t,\mu) &= \tilde{A}_4^{-1}(t)\tilde{A}_3(t) + O(\mu), \\ \tilde{H}(t,\mu) &= \left(\tilde{H}(t,\mu)\,k_2(t)c_2(t) + \tilde{A}_2(t)\right)A_4^{-1}(t) + O(\mu), \\ \tilde{H}(t,\mu) &= \tilde{A}_2(t)\tilde{A}_4^{-1}(t) + O(\mu). \end{aligned} \tag{A.4}$$

As a result of the decoupling transformation, the error dynamics system (A.1) will take the form of a system separated by time scales:

$$\begin{aligned} \dot{\varepsilon}_\xi(t) &= A_\xi(t,\mu)\varepsilon_\xi(t), \quad \varepsilon_\xi \in \mathbb{R}^{n_1}, \\ \mu\dot{\varepsilon}_\eta(t) &= A_\eta(t,\mu)\varepsilon_\eta(t), \quad \varepsilon_\eta \in \mathbb{R}^{n_2}, \quad t > t_0, \end{aligned} \tag{A.5}$$

where

$$A_\xi(t,\mu) = \tilde{A}_1(t) - \tilde{A}_2(t)\tilde{L}(t,\mu), \quad A_\eta(t,\mu) = \tilde{A}_4(t) + \mu\tilde{L}(t,\mu)\tilde{A}_2(t). \tag{A.6}$$

Moreover, according to Statement 1, the solutions (A.1) and (A.5) satisfy the following equalities:

$$\begin{aligned} \varepsilon_\xi(t) &= \varepsilon_x(t) + O(\mu),\ \varepsilon_x(t) = \varepsilon_\xi(t) + O(\mu), \\ \varepsilon_\eta(t) &= \tilde{A}_4^{-1}(t)\tilde{A}_3(t)\varepsilon_x(t) + \varepsilon_y(\tau) + O(\mu), \\ \varepsilon_y(t) &= -\tilde{A}_4^{-1}(t)\tilde{A}_3(t)\varepsilon_\xi(t) + \varepsilon_\eta(\tau) + O(\mu). \end{aligned} \tag{A.7}$$

Let's put

$$k_2(t) = k_f(t) \tag{A.8}$$

and we will look for $k_1(t)$ in the form:

$$k_1(t) = k_s(t) + \tilde{H}^0(t)k_2(t), \quad \tilde{H}^0(t) = (A_2(t) - k_s(t)c_2(t))A_4^{-1}(t). \tag{A.9}$$

Let's substitute (A.9), (A.8) into (A.6) and perform sequential transformations:

$$\begin{aligned} A_\xi(\mu) &\overset{(A.9)}{=} A_1 - \left(k_s + \tilde{H}^0 k_2\right) c_1 - \left(A_2 - \left(k_s + \tilde{H}^0 k_2\right) c_2\right) \tilde{L}(\mu) \\ &= \left(A_1 - A_2\tilde{L}(\mu)\right) - \left(k_s + \tilde{H}^0 k_2\right)\left(c_1 - c_2\tilde{L}(\mu)\right) \\ &\overset{(A.4)}{=} A_1 - A_2A_4^{-1}\left(k_2c_2\tilde{L}(\mu) + A_3 - k_2c_1\right) \\ &- \left(k_s + \tilde{H}^0k_2\right)\left[c_1 - c_2A_4^{-1}\left(k_2c_2\tilde{L}(\mu) + A_3 - k_2c_1\right)\right] + O(\mu) \\ &\overset{(As,cs)}{=} A_s - k_sc_s + \left(-A_2 + k_sc_2 + \tilde{H}^0k_2c_2\right)A_4^{-1}k_2c_2\tilde{L}(\mu) \\ &+ \left(A_2 - k_sc_2 - \tilde{H}^0A_4\right)A_4^{-1}k_2c_1 + \tilde{H}^0k_2c_2A_4^{-1}(A_3 - k_2c_1) + O(\mu) \\ &\overset{(A.9)}{=} A_s - k_sc_s + +\tilde{H}^0k_2c_2A_4^{-1}\left(A_4\tilde{L}(\mu) + k_2c_2\tilde{L}(\mu) - k_2c_1 + A_3\right) + O(\mu) \\ &\overset{(A.4)}{=} A_s - k_sc_s + O(\mu). \end{aligned}$$

Thus, for $A_\xi(t,\mu)$ with $k_1(t), k_2(t)$ of the form (A.9), (A.8) the approximation is valid:

$$A_\xi(t,\mu) = A_s(t) - k_s(t)c_s(t) + O(\mu). \tag{A.10}$$

Further, from (A.6) it follows

$$A_\eta(t,\mu) = (A_4(t) - k_2(t)c_2(t)) + O(\mu). \tag{A.11}$$

Thus, combining (A.10) and (A.11) from (A.5) we get:

$$\begin{aligned} \dot{\varepsilon}_\xi(t) &= (A_s(t) - k_s(t)c_s(t) + O(\mu))\,\varepsilon_\xi(t), \\ \mu\dot{\varepsilon}_\eta(t) &= (A_4(t) - k_2(t)c_2(t) + O(\mu))\,\varepsilon_\eta(t), \quad t > t_0. \end{aligned} \tag{A.12}$$

Since in (A.12) $k_s(t)$, $k_2(t)$ are the gain vectors for the observer (14) of the DS (8) and the observer (15) of the t-family of fast subsystems, then the parameters of the error system (A.12) are $O(\mu)$-close to the parameters of the error dynamics system for the DS and the t-family of fast subsystems observers with gain vectors k_s and k_f, respectively. Therefore, due to the continuous dependence of the solution (A.12) on additive perturbations of the system coefficients, the following estimates are valid: $||\varepsilon_\xi(t)|| \leqslant c_{\rho_s}\exp\left(-\rho(t-\bar{t})\right) + O(\mu)$, $t \geqslant \bar{t}$, $||\varepsilon_\eta(t)|| \leqslant c_{\rho_f}\exp\left(-\mu^0\rho\frac{(t-\bar{t})}{\mu}\right) + O(\mu)$, $t \geqslant \bar{t}$, whence, taking into account (A.7), it follows that the estimates are fair

$$\begin{aligned} &||\varepsilon_x(t)|| \leqslant c_{\rho_s}\exp(-\rho(t-\bar{t})) + O(\mu), \quad t \geqslant \bar{t}, \\ &||\varepsilon_y(t)|| \leqslant c_{\rho_s}||\tilde{A}_4^{-1}(t)\tilde{A}_3(t)||\exp(-\rho(t-\bar{t})) + c_{\rho_f}\exp\left(-\mu^0\rho\left(\frac{t-\bar{t}}{\mu}\right)\right) + O(\mu), \quad t \geqslant \bar{t}. \end{aligned}$$

Let $c_\rho = \max\{c_{\rho_s}, c_{\rho_f}, c_{\rho_s}||\tilde{A}_4^{-1}(t)\tilde{A}_3(t)||\}$. For $\mu \in (0, \mu^0]$ the estimate $\exp\left(-\mu^0\rho\left(\frac{t-\bar{t}}{\mu}\right)\right) < \exp\left(-\rho\left(t-\bar{t}\right)\right)$, $t \geqslant \bar{t}$, which implies the validity of the estimates $||\varepsilon(t,\mu)|| \leqslant c_\rho \exp(-\rho(t-\bar{t})) + O(\mu)$, $t \geqslant \bar{t}$, and, according to the Definition 7 and from the coincidence of (A.9) and (21) imply fairness of Theorem 1.

ACKNOWLEDGMENTS

I thank Professor A.I. Astrovskii for valuable advice and guidance expressed during the discussion of the material of this work.

REFERENCES

1. Kvakernaak, H. and Sivan, R., *Lineinye optimal'nye sistemy upravleniya* (Linear Optimal Control Systems), Moscow: Mir, 1977.

2. O'Reilly, J., *Observers for Linear Systems*, London: Academic Press, 1983.

3. Krasnova, S.A., Utkin, V., and Mikheev, Y.V., Cascade Design of State Observers, *Autom. Remote Control,* 2001, vol. 62, no. 2, pp. 207–226.

4. Korovin, S.K. and Fomichev, V.V., *Nablyudateli sostoyaniya dlya lineinykh sistem s neopredelennost'yu* (State Observers for Linear Systems with Uncertainty), Moscow: FIZMATLIT, 2007.

5. Astrovskii, A.I. and Gaishun, I.V., State Estimation for Linear Time-Varying Observation Systems, *Diff. Equat.,* 2019, vol. 55, no. 3, pp. 363–373.

6. Quoc, D. and Bobtsov, A., An Adaptive State Observer for Linear Time-varying Systems with Inaccurate Parameters, *Autom. Remote Control,* 2020, vol. 81, no. 12, pp. 2220–2229.

7. Luenberger, D.G., Observing the State of a Linear System, *IEEE Transactions on Military Electronics*, 1964, vol. 8., no. 2, pp. 74–80.

8. Luenberger, D.G., An Introduction to Observers, *IEEE Trans. Automat. Control*, 1971, vol. Ac-16, pp. 596–602.

9. Vasilyeva, A.B. and Dmitriev, M.G., Singular Perturbations in Optimal Control Problems, *Itogi Nauki i Tekhniki VINITI. Mat. Analiz,* 1982, vol. 20, pp. 3–77.

10. Naidu, D.S., Singular Perturbations and Time Scales in Control Theory and Applications: An Overview, *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms,* 2002, no. 9, pp. 233–278.

11. Dmitriev, M.G. and Kurina, G.A., Singular Perturbations in Control Problems, *Autom. Remote Control,* 2006, vol. 67, no. 1, pp. 1–43.

12. Zhang, Y., Naidu, D.S., Cai, C., and Zou, Y., Singular Perturbations and Time Scales in Control Theory and Applications: An overview 2002–1012, *Int. J. Inf. Sys. Sci.*, 2014, no. 9, pp. 1–36.

13. Kurina, G.A. and Kalashnikova, M.A., Singularly Perturbed Problems with Multi-Tempo Fast Variables, *Autom. Remote Control,* 2022, vol. 83, no. 11, pp. 1679–1723.

14. O'Reilly, J., Full-Order Observers for a Class of Ssingularly Perturbed Linear Time-Varying Systems, *Int. J. Control,* 1979, vol. 30, no. 5, pp. 745–756.

15. Yousfi, B., Raissi, T., Amairi, M., and Aoun, M., Interval Observers Design for Singularly Perturbed Systems, *Proc. 53rd IEEE Conf. Decision and Control, Los Angeles, CA, USA,* 2014, pp. 1637–1642.

16. Locatelli, A., State Observation and Output Feedback Stabilization of Linear Singularly Perturbable Systems, *IFAC Proceedings Volumes,* 1976, vol. 9, no. 3, pp. 335–343.

17. Porter, B., Singular Perturbation Methods in the Design of Full-Order Observers for Multivariable Linear Systems, *Int. J. Control,* 1977, vol. 26, no. 4, pp. 589–594.

18. Yoo, H. and Gajic, Z., New Designs of Reduced-Order Observer-Based Controllers for Singularly Perturbed Linear Systems, *Math. Probl. Eng.,* 2017, vol. 2017, pp. 1–14.

19. Duan, Z. and Kravaris, C., Reduced-order Nonlinear Observer Design for Two-time-scale Systems, *IFAC-PapersOnLine*, 2020, vol. 53, no. 2, pp. 5922–5927.

20. Deghat, M., Nesic, D., Teel, A.R., and Manzie, C., Observing the Slow States of General Singularly Perturbed Systems, *Proc. 59th IEEE Conf. Decision and Control (CDC), Jeju, Korea (South),* 2020, pp. 4206–4211.

21. Cuevas, L., Nesic, D., and Manzie, C., Robustness Analysis of Nonlinear Observers for the Slow Variables of Singularly Perturbed Systems, *Int. J. Robust Nonlinear Control,* 2020, vol. 30, no. 14, pp. 5628–5656.

22. Kalenova, V.I. and Morozov, V.M., *Lineinye nestatsionarnye sistemy i ikh prilozheniya k zadacham mekhaniki* (Linear Nonstationary Systems and their Applications to Problems of Mechanics), Moscow: FIZMATLIT, 2010.

23. Shin, D., On Quasi-Differential Operators in Hilbert Space, *Dokl. AN SSSR,* 1938, vol. 18, no. 5, pp. 523–526.

24. Astrovskii, A.I. and Gaishun, I.V., *Lineinye sistemy s kvazidifferentsiruemymi koeffitsientami: upravlyaemost' i nablyudaemost' dvizhenii* (Linear Systems with Quasi-Differentiable Coefficients: Controllability and Observability of Movements), Minsk: Belarus. Navuka, 2013.

25. Kopeikina, T.B. and Tsekhan, O.B., On the Theory of Observability of Linear Singularly Perturbed Systems, *Vesci NAN Belarusi. Ser. fiz.-mat. navuk,* 1999, no. 3, pp. 22–27.

26. Tsekhan, O.B., Conditions of Complete Observability of Linear time-invariant Singularly Perturbed System of the Second Order with Delay, *Vesn. GrDU imya Ya. Kupaly. Ser. 2, Matematyka. Fizika. Infarmatyka, vylichal'naya tekhnika i kiravanne,* 2014, no. 1 (170), pp. 53–64.

27. Tsekhan, O.B., Conditions for Pointwise Controllability and Pointwise Observability of Linear Time-Invariant Singularly Perturbed Systems with Delay, *Trudy Instituta Matematiki NAN Belarusi,* 2021, vol. 29, no. 1–2, pp. 138–148.

28. Tsekhan, O. and Pawluszewicz, E., Observability of Singularly Perturbed Linear Time-Varying Systems on Time Scales, *Proc. 26th Int. Conf. Methods and Models in Automation and Robotics (MMAR),* 2022, pp. 116–121.

29. Tsekhan, O.B., Robust Sufficient Conditions for Uniform Observability of a Linear Time-Varying Singularly Perturbed Systems, *Differencial'nye uravneniya i matematicheskaya fizika, Itogi nauki i tekhn. Ser. Sovrem. mat. i ee pril. Temat. obz., VINITI RAN, Moscow,* 2023. vol. 226, pp. 150–164.

30. Tsekhan, O., Robust Sufficient conditions for the observability of a linear time-invariant singularly perturbed system with delay, *Commun. Optim. Theory,* 2023, pp. 1–10.

31. Tsekhan, O.B., Quasidifferentiability and Uniform Observability of Linear Time-Varying Singularly Perturbed Systems, *Diff. Equat.,* 2023, vol. 59, no. 8, pp. 1130–1146.

32. Kokotovic, P.V., Khalil, H.K., and O'Reilly, J., Singular Perturbations Methods in Control: Analysis and Design, New York: Academic Press, 1999.

33. Chang, K., Singular Perturbations of a General Boundary Value Problem, *SIAM J. Math. Anal.,* 1972. vol. 3, no. 3, pp. 520–526.

34. Wolovich, W.A., On State Estimation of Observable Systems, *Preprint NASA Electronics Research Center. Cambridge,* 1968, no. 6, pp. 210–220.

35. Kalman, R.E., Falb, P.L., and Arbib, M.A., Topics in Mathematical System Theory, New York: McGraw-Hill, 1969.

36. Astrovskii, A.I., *Nablyudaemost' lineinykh nestatsionarnykh sistem* (Observability of Linear Time-Varying Systems), Minsk: MIU, 2007.

37. Gaishun, I.V., *Vvedenie v teoriyu lineinyh nestatsionarnykh sistem* (Introduction to the Theory of Linear Time-Varying Systems), Minsk: Institut matematiki NAN Belarusi, 1999.

This paper was recommended for publication by A.A. Bobtsov, a member of the Editorial Board